Some examples in the integral and Brown-Peterson cohomology of p-groups.

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Introduction.

For a finite group G, we define the Chern ring, $\mathrm{Ch}(G)$, to be the subring of $H^{\mathrm{even}}(G;\mathbb{Z})$ generated by Chern classes of representations of G. We say that G has p-rank n if n is maximal such that G contains a subgroup isomorphic to $(C_p)^n$. In [3] Atiyah showed that for any finite group G, $K^0(\mathrm{B}G)$ is the completion of the representation ring of G with respect to a certain topology. The filtration of $K^0(\mathrm{B}G)$ given by the E_∞ page of the Atiyah-Hirzebruch spectral sequence (AHSS) gives rise to a filtration of the representation ring of G. Atiyah conjectured that this filtration coincided with another filtration defined algebraically, and remarked that this conjecture is equivalent to the conjecture that $\mathrm{Ch}(G)$ maps onto the E_∞ page of the AHSS. (It is clear that $\mathrm{Ch}(G)$ consists of universal cycles because the AHSS for $\mathrm{B}U(n)$ collapses.) Weiss discovered that the alternating group A_4 gives a counterexample to this conjecture [16], and Thomas has exhibited many counterexamples, all of which have order divisible by more than one prime [14].

Thomas showed that the split metacyclic p-groups and various other p-groups of p-rank two have the property that the Chern subring is the whole of the even degree integral cohomology, and conjectured that this property would hold for all p-groups of p-rank two [12], [13], [15]. The group A_4 shows that the conjecture cannot be extended to groups of non-prime power order. AlZubaidy claimed to have verified this conjecture, but some of his proofs are flawed [1], [2]. Recently Huebschmann and Tezuka-Yagita have shown that

$\mathrm{Ch}(G) = H^{\mathrm{even}}(G;\mathbb{Z})$ for any metacyclic p-group G [6], [11]. For $p \geq 5$ Blackburn's classification [4] implies that the only p-groups of p-rank two not covered by the above theorems are the groups $G(n,\epsilon)$ defined below. We show that $\mathrm{Ch}(G(n,\epsilon))$ is strictly contained in $H^{\mathrm{even}}(G(n,\epsilon);\mathbb{Z})$ for each such group. These groups also afford p-group counterexamples to the conjecture of Atiyah described above.

Similar calculations may be made in the Brown-Peterson cohomology rings of these groups. These enable us to give a negative answer to a question of Landweber [7], who asked if Chern classes generate the Brown-Peterson cohomology of every p-group.

The examples.

The groups which we shall consider may be presented as

$$G(n,\epsilon) = \langle A, B, C \mid A^p = B^p = C^{p^{n-2}} = [B,C] = 1 \quad [A, C^{-1}] = B \quad [B,A] = C^{\epsilon p^{n-3}} \rangle,$$

where p is a prime not equal to 2 or 3, $n \geq 4$, and for fixed p and n there are two isomorphism classes of such groups, depending whether ϵ is either 1 or a quadratic non-residue modulo p. The group $G(n,\epsilon)$ has order p^n. In the sequel we shall refer to G instead of $G(n,\epsilon)$ unless the values of n and ϵ are important. The subgroup M generated by B and C is maximal (and hence normal) and is isomorphic to $C_p \oplus C_{p^{n-2}}$. We define one dimensional representations θ and ϕ of the group M by

$$\theta : B^j C^k \mapsto \exp(2\pi i j/p)$$

$$\phi : B^j C^k \mapsto \exp(2\pi i k/p^{n-2}).$$

The action of the quotient group G/M on the representation ring of M is that conjugation by A sends θ to $\theta \otimes \phi^{\otimes p^{n-3}}$ and sends ϕ to $\phi \otimes \theta^{\otimes \epsilon}$. Later we shall define elements of $H^2(M;\mathbb{Z})$ and $BP^2(\mathrm{B}M)$ as Chern classes, and the action of G/M on these elements will be determined by its action on the representations θ and ϕ.

The group G has only 1- and p-dimensional irreducible representations because it has an abelian subgroup (M in fact) of index p. A one dimensional representation of G must

restrict trivially to $\langle B\rangle$, and a p-dimensional representation of G restricts to $\langle B\rangle$ as either p copies of the same representation of $\langle B\rangle$, or as the sum of one copy of each of the one-dimensional representations of $\langle B\rangle$. The examples $\mathrm{Ind}_M^G(\theta)$ and $\mathrm{Ind}_M^G(\phi)$ show that both these alternatives do occur.

Now define generators β, γ for $H^2(M;\mathbb{Z})$ by

$$\beta = c_1(\theta) \qquad \gamma = c_1(\phi),$$

$$\text{so that} \qquad H^{\mathrm{even}}(M;\mathbb{Z}) \cong \mathbb{Z}[\beta,\gamma]/(p\beta, p^{n-2}\gamma),$$

and let β' be the restriction to $\langle B\rangle$ of β, so that

$$H^*(\langle B\rangle;\mathbb{Z}) \cong \mathbb{Z}[\beta']/(p\beta').$$

Lemma 1. *With notation as above, the image of* $\mathrm{Ch}(G)$ *under restriction to* $\langle B\rangle$ *is the subring of* $H^*(\langle B\rangle;\mathbb{Z})$ *generated by* β'^{p-1} *and* β'^p. *For all* $m \geq 0$,

$$\beta'^{m+p-1} = -\mathrm{Res}^G_{\langle B\rangle}\mathrm{Cor}^G_M(\gamma^{p-1}\beta^m).$$

Proof. If ρ is a 1-dimensional representation of G then its Chern class restricts trivially to $\langle B\rangle$. If ρ is a p-dimensional representation of G, then either ρ restricts to $\langle B\rangle$ as p-copies of the same representation, in which case

$$\mathrm{Res}(c.(\rho)) = (1+i\beta')^p = 1 + i\beta'^p,$$

or as one copy of each representation, in which case

$$\mathrm{Res}(c.(\rho)) = \prod_{i=0}^{p-1}(1+i\beta') = 1 - \beta'^{p-1}.$$

By applying the double coset formula we see that

$$\begin{aligned}
\mathrm{Res}^G_{\langle B\rangle}\mathrm{Cor}^G_M(\gamma^{p-1}\beta^m) &= \mathrm{Res}^M_{\langle B\rangle}(\sum_{i=0}^{p-1} c^*_{A^i}(\gamma^{p-1}\beta^m)) \\
&= \mathrm{Res}^M_{\langle B\rangle}(\sum_{i=0}^{p-1}(\gamma + i\epsilon\beta)^{p-1}(\beta + ip^{n-3}\gamma)^m) \\
&= -\beta'^{m+p-1}
\end{aligned}$$

■

Remarks. The image of $\mathrm{Res}^G_{\langle B\rangle}$ is precisely the subring of $H^*(\langle B\rangle;\mathbb{Z})$ generated by β'^{p-1}, β'^{p}, $\beta'^{p+1},\ldots,\beta'^{2p-3}$. One way to show this is by considering the subgroup N of G generated by A and B. This subgroup is normal in G, and is the non-abelian group of order p^3 and exponent p. Using Lewis' calculation of $H^*(N;\mathbb{Z})$ [9], it may be shown that the image of $H^*(N;\mathbb{Z})^{G/N}$ under restriction to $\langle B\rangle$ does not contain β'^i for $i<p-1$.

Corollary 2. $\mathrm{Ch}(G)$ *is strictly contained in* $H^{\mathrm{even}}(G;\mathbb{Z})$. *Moreover,* $\mathrm{Ch}(G)$ *does not map onto* $H^*(G;\mathbb{Z})$ *modulo its nilradical.*

Proof. We know that β'^{p+1} is in $\mathrm{Res}^G_{\langle B\rangle}(H^{\mathrm{even}}(G;\mathbb{Z}))$, but not in $\mathrm{Res}^G_{\langle B\rangle}(\mathrm{Ch}(G))$. ∎

Corollary 3. *In the AHSS for G, write $B_\infty(G)$ for the universal boundaries, and $Z_\infty(G)$ for the universal cycles. Then* $\mathrm{Ch}(G)+B_\infty(G)$ *is strictly contained in* $Z_\infty(G)$.

Proof. The AHSS for $\langle B\rangle$ collapses, so $\mathrm{Res}^G_{\langle B\rangle}(B_\infty(G))$ is trivial. Corestrictions of Chern classes are universal cycles, so $\beta'^{p+1}\in\mathrm{Res}^G_{\langle B\rangle}(Z_\infty(G))$, but $\beta'^{p+1}\notin\mathrm{Res}^G_{\langle B\rangle}(\mathrm{Ch}(G)+B_\infty(G))$. ∎

For any generalised cohomology theory $\mathcal{H}$ and any group K, we may define $\mathrm{Ch}_{\mathcal{H}}(K)$ to be the subring of $\mathcal{H}^*(\mathrm{B}K)$ generated by $\rho^*(\mathcal{H}^*(\mathrm{B}U))$ for all representations ρ of K in a unitary group U. We now give a result analogous to Corollary 2 for Brown-Peterson cohomology.

Lemma 4. $\mathrm{Ch}_{BP}(G)$ *is strictly contained in* $BP^*(\mathrm{B}G)$.

Proof. As in the integral cohomology case, define elements β and γ in $BP^2(M)$ by $\beta=c_1(\theta)$, $\gamma=c_1(\phi)$, and also define $\beta'=\mathrm{Res}^M_{\langle B\rangle}(\beta)$, so that

$$BP^*(\mathrm{B}M)\cong BP_*[[\beta,\gamma]]/([p]\beta,[p^{n-2}]\gamma),\qquad BP^*(\mathrm{B}\langle B\rangle)\cong BP_*[[\beta']]/([p]\beta'),$$

where $[r]x$ stands for the BP formal group sum of r copies of x. Let '$\equiv$' stand for congruence modulo the ideal of $BP^*(\mathrm{B}\langle B\rangle)$ generated by p, v_1, $v_2,\ldots$. As in Lemma 1, if ρ is a

p-dimensional representation of G, then either

$$\begin{aligned} \operatorname{Res}^G_{\langle B\rangle}(c.(\rho)) &= (1+[i]\beta')^p \equiv 1+i\beta'^p \\ \text{or}\quad \operatorname{Res}^G_{\langle B\rangle}(c.(\rho)) &= \prod_{i=0}^{p-1}(1+[i]\beta') \equiv 1-\beta'^{p-1}. \end{aligned}$$

Also, we have that

$$\begin{aligned} \operatorname{Res}^G_{\langle B\rangle}\operatorname{Cor}^G_M(\gamma^{p-1}\beta^2) &= \operatorname{Res}^M_{\langle B\rangle}(\sum_{i=0}^{p-1}(\gamma +_{BP} [i\epsilon]\beta)^{p-1}(\beta +_{BP} [ip^{n-3}]\gamma)^2 \\ &\equiv -\beta'^{p+1}. \end{aligned}$$ ■

Our original proofs of these results involved calculation with $BP^*(\mathrm{B}N)$, which has been determined by Tezuka-Yagita [10], and with the integral cohomology of the non-abelian maximal subgroups of G, determined by Leary [8]. Using these methods we obtain more information concerning $BP^*(\mathrm{B}G)$ and $H^*(G;\mathbb{Z})$, which we intend to publish later.

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